



## Common Fixed Points for Four Self-Maps in Cone Metric Spaces

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### ABSTRACT

We prove, the existence of coincidence points and common fixed points for four self-mappings satisfying a generalized contractive condition without normal cone in cone metric spaces. Our results generalize several well-known recent and classical results.

**Keywords:** Coincidence points; Common fixed point; (IT)-Commuting; Cone metric space.

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## INTRODUCTION AND PRELIMINARIES

The study of common fixed points of mappings satisfying certain contractive conditions has been at the centre of vigorous research activity, being the applications of fixed point very important in several areas of Mathematics. In 2007 Huang and Zhang [12] have generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractive conditions. Subsequently, Abbas and Jungck[1] and Abbas and Rhoades [2], Ilic and Rakocevic [13], Rezapour and Hambarani[14] and Vetro [17] studied fixed point theorems for contractive type mappings in cone metric spaces .

The aim of this paper is to obtain points of coincidence and common fixed points for four self-mappings satisfying generalized contractive type condition without normal in cone metric spaces. Our result generalized and extends several existing fixed point theorems in the literature.

In all that follows  $B$  is a real Banach Space, and  $\theta$  denotes the zero element of  $B$ .

For the mapping  $f, g: X \rightarrow X$ , let  $C(f, g)$  denote the set of coincidence points of  $f$  and  $g$ , that is  $C(f, g) = \{z \in X : fz = gz\}$ .

The following definitions are due to Huang and Zhang [12].

**Definition 1.1.** Let  $B$  be a real Banach Space and  $P$  a subset of  $B$ . The set  $P$  is called a cone if and only if:

- (a).  $P$  is closed, non –empty and  $P \neq \{\theta\}$ ;
- (b).  $a, b \in P$ ,  $a, b \geq 0$ ,  $x, y \in P$  implies  $ax+by \in P$ ;
- (c).  $x \in P$  and  $-x \in P$  implies  $x = \theta$ .

**Definition 1.2.** Let  $P$  be a cone in a Banach Space  $B$ , define partial ordering ' $\leq$ ' with respect to  $P$  by  $x \leq y$  if and only if  $y - x \in P$ . We shall write  $x < y$  to indicate  $x \leq y$  but  $x \neq y$  while  $x \ll y$  will stand for  $y - x \in \text{Int } P$ , where  $\text{Int } P$  denotes the interior of the set  $P$ . This Cone  $P$  is called an order cone.

**Definition 1.3.** Let  $B$  be a Banach Space and  $P \subset B$  be an order cone. The order cone  $P$  is called normal if there exists  $L > 0$  such that for all  $x, y \in B$ ,

$$\theta \leq x \leq y \text{ implies } \|x\| \leq L \|y\|.$$

The least positive number  $L$  satisfying the above inequality is called the normal constant of  $P$ .

**Definition 1.4.** Let  $X$  be a nonempty set of  $B$ . Suppose that the map  $d: X \times X \rightarrow B$  satisfies:

$$(d1). \theta \leq d(x, y) \text{ for all } x, y \in X \text{ and}$$

$$d(x, y) = \theta \text{ if and only if } x = y;$$

$$(d2). d(x, y) = d(y, x) \text{ for all } x, y \in X;$$

$$(d3). d(x, y) \leq d(x, z) + d(y, z) \text{ for all } x, y, z \in X.$$

Then  $d$  is called a cone metric on  $X$  and  $(X, d)$  is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space.

**Example 1.5.** ([12]). Let  $B = \mathbb{R}^2$ ,  $P = \{(x, y) \in B \text{ such that } : x, y \geq 0\} \subset \mathbb{R}^2$ ,  $X = \mathbb{R}$  and  $d: X \times X \rightarrow B$  such that  $d(x, y) = (\|x - y\|, \alpha \|x - y\|)$ , where  $\alpha \geq 0$  is a constant. Then  $(X, d)$  is a cone metric space.

**Definition 1.6.** Let  $(X, d)$  be a cone metric space. We say that  $\{x_n\}$  is

- (i) a Cauchy sequence if for every  $c$  in  $B$  with  $c \gg \theta$ , there is  $N$  such that for all  $n, m > N$ ,  $d(x_n, x_m) \ll c$ ;
- (ii) Convergent sequence if for any  $c \gg \theta$ , there is an  $N$  such that for all  $n > N$ ,  $d(x_n, x) \ll c$ , for some fixed  $x$  in  $X$ . We denote this  $x_n \rightarrow x$  (as  $n \rightarrow \infty$ ).

**Lemma 1.7.** Let  $(X, d)$  be a cone metric space, and let  $P$  be a normal cone with normal constant  $L$ .

Let  $\{x_n\}$  be a sequence in  $X$ . Then

- (i).  $\{x_n\}$  converges to  $x$  if and only if  $d(x_n, x) \rightarrow 0$  ( $n \rightarrow \infty$ ).
- (ii).  $\{x_n\}$  is a Cauchy sequence if and only if  $d(x_n, x_m) \rightarrow 0$  (as  $n, m \rightarrow \infty$ ).

## 2. Common Fixed Point Theorem



In this section we prove, the existence of coincidence points and common fixed points for four self-mappings satisfying a generalized contractive condition without normal cone in cone metric spaces. Our result generalizes and extends the results of A.Azam et al.[7] .

The following Theorem is extend and improve the Theorem 1 of A.Azam et al.[7] .

**Theorem 2.1:** Let  $(X, d)$  be a cone metric space. Suppose the mappings  $S, T, f, g: X \rightarrow X$  satisfy

$$d(Sx, Ty) \leq Ad(fx, gy) + B[d(fx, Sx) + d(gy, Ty)] + C[d(fx, Ty) + d(gy, Sx)] \quad \dots (1)$$

for all  $x, y \in X$ , where  $A, B, C$  are non negative real numbers with  $A+2B+2C < 1$ .

If  $T(x) \subseteq f(x)$ ,  $S(x) \subseteq g(x)$  and one of  $f(x)$ ,  $g(x)$ ,  $S(x)$  or  $T(x)$  is a complete subspace of  $X$ , then

$\{S, f\}$  and  $\{T, g\}$  have a coincidence point in  $X$ . Moreover, if  $\{S, f\}$  and  $\{T, g\}$  are (IT)- Commuting, then  $S, T, f$  and  $g$  have a unique common fixed point.

**Proof:** Suppose  $x_0$  is an arbitrary point of  $X$  and define the sequence  $\{y_n\}$  in  $X$  such that

$$y_{2n} = Sx_{2n} = gx_{2n+1}$$

$$y_{2n+1} = Tx_{2n+1} = fx_{2n+2}$$

By (1) we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\leq A d(fx_{2n}, gx_{2n+1}) + B[d(fx_{2n}, Sx_{2n}) + d(gx_{2n+1}, Tx_{2n+1})] \\ &\quad + C[d(fx_{2n}, Tx_{2n+1}) + d(gx_{2n+1}, Sx_{2n})] \\ &\leq A d(y_{2n-1}, y_{2n}) + B[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] \\ &\quad + C[d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})] \\ &\leq A d(y_{2n-1}, y_{2n}) + B[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] \\ &\quad + C[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] \\ &\leq (A+B+C) d(y_{2n-1}, y_{2n}) + (B+C) d(y_{2n}, y_{2n+1}) \end{aligned}$$

which implies that

$$d(y_{2n}, y_{2n+1}) \leq \frac{A+B+C}{1-(B+C)} d(y_{2n-1}, y_{2n})$$

(or)

$$d(y_{2n}, y_{2n+1}) \leq \lambda d(y_{2n-1}, y_{2n})$$

$$\text{where, } \lambda = \frac{A+B+C}{1-(B+C)} < 1.$$

Similarly, it can be shown that

$$d(y_{2n+1}, y_{2n+2}) \leq \lambda d(y_{2n}, y_{2n+1}).$$

Therefore, for all  $n$ ,

$$d(y_{n+1}, y_{n+2}) \leq \lambda d(y_n, y_{n+1}) \leq \dots \leq \lambda^{n+1} d(y_0, y_1).$$

Now, for any  $m > n$ ,

$$\begin{aligned} d(y_n, y_m) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \dots + d(y_{m-1}, y_m) \\ &\leq [\lambda^n + \lambda^{n+1} + \dots + \lambda^{m-1}] d(y_0, y_1) \\ &\leq \frac{\lambda^m}{1-\lambda} d(y_0, y_1). \end{aligned}$$

Let  $0 < c < 1$  be given.

Choose  $\delta > 0$  such that  $c + \{x \in Z : \|x\| < \delta\} \subseteq P$ .



Also choose a natural number  $N_1$  such that  $\frac{\lambda^m}{1-\lambda} d(y_0, y_1) \in \{x \in Z: \|x\| < \delta\}$ , for all  $m \geq N_1$ .

Then,  $\frac{\lambda^m}{1-\lambda} d(y_0, y_1) < c$ , for all  $m \geq N_1$ .

Thus  $n > m$ ,  $d(y_n, y_m) \leq \frac{\lambda^m}{1-\lambda} d(y_0, y_1) < c$ ,

which implies that  $\{y_n\}$  is a Cauchy sequence.

Since  $\{y_n\}$  is a Cauchy Sequence in  $T(X)$  which is complete, there exists  $z \in T(X)$  such that  $y_n \rightarrow z$ .

Since  $T(X) \subset f(X)$ , then there exist a point  $u \in X$  such that  $z = fu$ .

Let us prove that  $z = Su$ .

Choose a natural number  $N_2$  such that for all  $n \geq N_2$

$$\begin{aligned} d(y_{2n-2}, y_{2n-1}) &< \left[ \frac{c(1-B-C)}{3B} \right], \\ d(y_{2n-2}, z) &< \left[ \frac{c(1-B-C)}{3(A+C)} \right], \\ \text{and } d(y_{2n-1}, z) &< \left[ \frac{c(1-B-C)}{3(1+C)} \right]. \end{aligned}$$

Now inequality (1) implies that (By the triangle inequality)

$$\begin{aligned} d(fu, Su) &\leq d(fu, Tx_{2n-1}) + d(Tx_{2n-1}, Su) \\ &\leq d(z, Tx_{2n-1}) + d(Su, Tx_{2n-1}) \\ &\leq d(z, Tx_{2n-1}) + Ad(fu, gx_{2n-1}) + B[d(fu, Su) + d(gx_{2n-1}, Tx_{2n-1})] + C[d(fu, Tx_{2n-1}) + d(gx_{2n-1}, Su)] \\ &\leq d(z, Tx_{2n-1}) + Ad(z, gx_{2n-1}) + B[d(fu, Su) + d(gx_{2n-1}, Tx_{2n-1})] + C[d(z, Tx_{2n-1}) + d(gx_{2n-1}, Su)] \\ &\leq d(z, y_{2n-1}) + Ad(z, y_{2n-2}) + B[d(fu, Su) + d(y_{2n-2}, y_{2n-1})] + C[d(z, y_{2n-1}) + d(y_{2n-2}, Su)] \\ &\leq d(z, y_{2n-1}) + Ad(z, y_{2n-2}) + B[d(fu, Su) + d(y_{2n-2}, y_{2n-1})] + C[d(z, y_{2n-1}) + d(y_{2n-2}, fu)] + d(fu, Su) \\ &\leq (1+C) d(z, y_{2n-1}) + (A+C)d(z, y_{2n-2}) + Bd(y_{2n-2}, y_{2n-1}) + (B+C)d(fu, Su) \end{aligned}$$

$$d(fu, Su) - (B+C)d(fu, Su) \leq (1+C) d(z, y_{2n-1}) + (A+C)d(z, y_{2n-2}) + Bd(y_{2n-2}, y_{2n-1})$$

$$[1-(B+C)]d(fu, Su) \leq (1+C) d(z, y_{2n-1}) + (A+C)d(z, y_{2n-2}) + Bd(y_{2n-2}, y_{2n-1})$$

$$d(fu, Su) \leq \left[ \frac{1+C}{(1-B-C)} \right] d(z, y_{2n-1}) + \left[ \frac{A+C}{(1-B-C)} \right] d(z, y_{2n-2}) + \left[ \frac{B}{(1-B-C)} \right] d(y_{2n-2}, y_{2n-1})$$

$$\begin{aligned} &< \left[ \frac{1+C}{(1-B-C)} \right] \left[ \frac{c(1-B-C)}{3(1+C)} \right] + \left[ \frac{A+C}{(1-B-C)} \right] \left[ \frac{c(1-B-C)}{3(A+C)} \right] + \left[ \frac{B}{(1-B-C)} \right] \\ &\quad + \left[ \frac{c(1-B-C)}{3B} \right] \end{aligned}$$

$$< \frac{c}{3} + \frac{c}{3} + \frac{c}{3}.$$



Thus,  $d(fu, Su) < \frac{c}{3}$ , for  $1 \leq m \leq \infty$ .

So,  $\frac{c}{3} - d(fu, Su) \in P$ , for all  $m \geq 1$ .

Since  $\frac{c}{3} \rightarrow 0$  as  $m \rightarrow \infty$  and  $P$  is closed,

$- d(fu, Su) \in P$ . But  $P \cap (-P) = \{0\}$ .

Therefore,  $d(fu, Su) = 0$ .

Hence,  $z = fu = Su$ ;  $u$  is a coincidence point of  $\{S, f\}$ .

... (2)

Since,  $S(X) \subseteq g(X)$  there exists a point  $v \in X$  such that  $z = gv$ . We shall show that  $Tv = z$ .

Then by (1), we have

$$\begin{aligned} d(z, Tv) &= d(Su, Tv) \\ &\leq A d(fu, gv) + B[d(fu, Su) + d(gv, Tv)] + C[d(fu, Tv) + d(gv, Su)], \\ &\leq A d(z, z) + B[d(z, z) + d(z, Tv)] + C[d(z, Tv) + d(z, z)], \\ &\leq (B+C) d(z, Tv), \end{aligned}$$

which is a contradiction. Since  $A + 2B + 2C < 1$ .

Implies  $z = Tv$ .

Therefore,  $z = Tv = gv$ ,  $v$  is a coincidence point of  $\{T, g\}$ .

... (3)

From (2) and (3) it follows that

$$Su = fg = Tv = gv (=z).$$

... (4)

Since  $(S, f)$ ,  $(T, g)$  are (IT) – Commuting.

$$\begin{aligned} d(SSu, Su) &= d(SSu, fu) \\ &= d(SSu, Tv) \\ &\leq A d(fSu, gv) + B[d(fSu, SSu) + d(gv, Tv)] + C[d(fSu, Tv) + d(gv, SSu)], \\ &= A d(Sfu, gv) + B[d(Sfu, SSu) + d(gv, Tv)] + C[d(Sfu, Tv) + d(gv, SSu)], \\ &= A d(SSu, Su) + B[d(SSu, SSu) + d(z, z)] + C[d(SSu, Su) + d(Su, SSu)], \\ &\leq [A + B + 2C] d(SSu, Su), \text{ which is a contradiction, Since } A + 2B + 2C < 1. \end{aligned}$$

Therefore,  $SSu = Su (=z)$ .

$$Su = SSu = Sfu = fSu \quad (\text{since, } (S, f) \text{ is (IT)- Commuting})$$

$$\Rightarrow SSu = fSu = Su (=z).$$

Therefore,  $Su (=z)$  is a common fixed point of  $S$  and  $f$ .

... (5)

Similarly,  $Tv = TTv = Tgv = gTv$ ,

$$\Rightarrow TTv = STv = Tv (=z).$$

Therefore,  $Tv (=z)$  is a common fixed point of  $T$  and  $g$ .

... (6)

In view of (5) and (6), it follows that  $S, T, f$  and  $g$  have a common fixed point namely  $z$ .

Uniqueness, let  $W$  be another common fixed point of  $S, T, f$  and  $g$ , then

$$\begin{aligned} d(z, w) &= d(Sz, Tw) \\ &\leq A d(fz, gw) + B[d(fz, Sz) + d(gw, Tw)] + C[d(fz, Tw) + d(gw, Sz)], \\ &\leq A d(z, w) + B[d(z, z) + d(w, w)] + C[d(z, w) + d(w, z)], \\ &= A d(z, w) + 2C d(z, w), \\ &\leq (A + 2C) d(z, w), \text{ which is a contradiction, since } A + 2B + 2C < 1. \end{aligned}$$





Therefore,  $z = w$ .

Hence,  $z$  is a unique common fixed point of  $S$ ,  $T$ ,  $f$  and  $g$  respectively.

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